On the renormalisation group transformations for the linear Z(3) model with a magnetic field

1986 J. Phys. A: Math. Gen. 19 259

(http://iopscience.iop.org/0305-4470/19/2/019)

J. Phys. A: Math. Gen. 19 (1986) 259-265. Printed in Great Britain

On the renormalisation group transformations for the linear $Z(3)$ model with a magnetic field

J C Cressoni and R J V dos Santos

Departamento de Física, Universidade Federal de Alagoas, 57000 Maceió, Brazil

Received 9 May 1985

Abstract. The linear $Z(3)$ spin system in the presence of a magnetic field is studied by a dedecoration renormalisation group (RG) transformation. In order to perform the transformation, non-usual terms must be added to the Hamiltonian. The linear region around a particular $T=0$ fixed point, where these terms go to zero, is investigated and the 'critical behaviour' of the free energy, correlation function and susceptibility is obtained. The system is solved exactly and the results are compared with those obtained by the RG approach. Magnetisation and susceptibility curves against temperature are also shown.

1. Introduction

An example of an exact renormalisation group (RG) transformation was given by Nelson and Fisher (1975) by performing dedecoration group transformations in the one-dimensional Ising model in the presence of an external magnetic field. The same procedures can be easily generalised to study a one-dimensional $Z(N)$ spin system in zero magnetic field (Cressoni 1981). If we introduce a magnetic field, however, we note that to treat these models via RG theory one has to augment the dimensionality of their parameter spaces to make them closed by RG operations (i.e. having no different operators in the iterated Hamiltonians). This occurs just for $N>2$ since the $Z(2)$ is the familiar Ising model whose Hamiltonian is already sufficiently general to be treated by the RG formalism. In this way, the usual $Z(N)$ Hamiltonian (Alcaraz and Köberle 1981)

$$\mathscr{H} = -\sum_{\langle i,j\rangle} \sum_{m=1}^{\bar N} \tfrac{1}{2} J_m[(S_i S_j^+)^m + \mathrm{cc}] + \mathrm{ZT} \tag{1.1a}$$

$$= -\sum_{\langle i,j\rangle} \sum_{m=1}^{\bar N} J_m\left(\cos\frac{2\pi m(n_i - n_j)}{N}\right) + \mathrm{ZT} \tag{1.1b}$$

where

(i) $S_k^N = 1 \Rightarrow S_k = \exp(2\pi \mathrm{i} n_k/N)$, $n_k = 0, 1, 2, \ldots, N-1$;
(ii) $\bar N$ is the largest integer smaller or equal to $\frac{1}{2}N$;
(iii) ZT stands for Zeemann terms;
(iv) $\langle\ \rangle$ indicates sum over nearest neighbours,

must be generalised to guarantee that the iterated Hamiltonian will have a similar form. This can be done by modifying slightly the interaction in (1.1a) to $J_m(S_i S_j^+)^m + J_m^*(S_i^+ S_j)^m$, thus permitting the coupling constants to be complex (which introduces sine-like terms) and by considering new (anisotropic-like) terms of the form

$$\tilde J_{lm} S_i^l (S_j^+)^m + \tilde J_{lm}^* (S_i^+)^l S_j^m \qquad l \neq m \tag{1.2}$$

which, as the ZT, break the global $Z(N)$ symmetry ($n_i \to n_i + \Delta$, mod N). The resulting Hamiltonian will certainly be general enough to avoid the difficulty outlined above.

We realised that the $N=3$ case did not need complex coupling constants which reduces considerably the dimension of the parameter space. The $N=3$ real coupling constants Hamiltonian can be written as

$$\mathscr{H} = -\sum_{\langle i,j\rangle}\left[J\left(\cos\frac{2\pi(n_i-n_j)}{3}-1\right)+\tilde{J}\left(\cos\frac{2\pi(n_i+n_j)}{3}-1\right)\right]$$
$$-\sum_i H(\delta n_{i,0}-1)-\mathscr{N}(I+J+\tilde{J}+H) \qquad (1.3)$$

or

$$\mathscr{H} = -\sum_{\langle i,j\rangle}\left(\frac{3J}{2}(\delta_{ni,nj}-1)+\frac{3J}{2}(\delta_{ni,3-nj}-1)\right)-\sum_i H(\delta_{ni,0}-1)-\mathscr{N}(I+J+\tilde{J}+H) \qquad (1.4)$$

where I is a constant or spin independent term and $\mathscr{N}$ is the number of degrees of freedom. The first Kronecker delta shows that the $Z(3)$ model (i.e. (1.4) with $\tilde{J}=0$) is the scalar (or vector) 3-state Potts model. In fact, it is an easy task to verify that the $Z(N)$ Hamiltonian (1.1a) generalises the scalar (and vector) N-state Potts model (Alcaraz and Köberle 1981).

When $J=\tilde{J}$ this Hamiltonian displays a non-local symmetry of the following type: it remains unchanged under reflection of alternating spins through the $n=0$ axis, i.e. $0\to 0$, $1\to 2$ and $2\to 1$. Since the usual magnetisation order parameter $\langle\frac{1}{2}(S_i+S_i^+)\rangle = \langle\cos(2\pi n_i/3)\rangle$ also displays such a symmetry, it can take on non-zero values at finite temperatures. Another order parameter has to be used if one wants to test whether the system respects the symmetry of the Hamiltonian in accordance with the Mermin-Wagner theorem or not. We will not go on further discussing the system described by (1.4) since our interest resides mainly in the study of regions where $\tilde{J}=0$, i.e. where the usual Potts model is recovered.

2. Renormalisation group transformation

The dedecoration group we have used was defined by Fisher (1959) and applied to the linear Ising chain by Nelson and Fisher (1975). Given a set of three consecutive spins in the chain, we replace the central spin by a single bond joining the two external spins (see figure 1). The magnetic fields acting on the remaining spins s_1 and s_3 are changed by the quantities δL_1 and δL_3 ($L = H/(K_B T)$) and the set of interactions $\{K\}$ (with $K = J/(K_B T)$) is changed to $\{K'\}$. By dedecorating every other spin we generate a RG transformation with spatial and spin rescaling factors $b=2$ and $c=1$, respectively. In terms of the new variables

$$x=\exp(-3K/2) \qquad y=\exp(-L/2) \qquad z=\exp(-3\tilde{K}/2) \qquad (2.1)$$

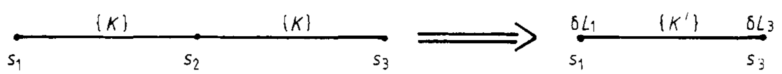

Figure 1. The dedecoration transformation. R represents the RG operator, $\{K\}$ is the set of interactions and δL is the amount by which the magnetic field has to be changed.

one gets the following recursion relations:

$$x' = xz\left(\frac{(xz+2y^2)(1+xy^2+zy^2)^2}{(x^2z^2+y^2z^2+x^2y^2)^2(1+2x^2y^2z^2)}\right)^{1/3} \tag{2.2}$$

$$y' = y\left(\frac{(x^2z^2+y^2z^2+x^2y^2)(xz+2y^2)}{(1+xy^2+zy^2)(1+2x^2y^2z^2)}\right)^{1/3} \tag{2.3}$$

$$z' = \left(\frac{(x^2z^2+y^2z^2+x^2y^2)(1+xy^2+zy^2)^2}{(xz+2y^2)^2(1+2x^2y^2z^2)}\right)^{1/3}. \tag{2.4}$$

The spin-independent term is renormalised according to

$$\phi_2 = 1+2x^2y^2z^2 \tag{2.5}$$

where ϕ_2 is a new variable defined as $\phi_2 = \exp(C'+K'+\tilde{K}'+L')/\exp[2(C+K+\tilde{K}+L)]$ with C standing for $I/(K_B T)$.

In the unit cube $(J>0, \tilde{J}>0)$ the following fixed points may be found (see figure 2): an attractive line of fixed points $x^* = z^* = 1$ (independent of y) corresponding to $T=\infty$; a 'frozen' fixed point $(x^*, y^*, z^*) = (0,0,0)$; and a 'ferromagnetic' fixed point at $(x^*, y^*, z^*) = (0,1,1)$ corresponding to $T=0$, $\tilde{J}=0$, $H=0$. This is the point in which we have an interest, since it implies $\tilde{J}=0$ (meaning that the usual Potts model is recovered). One also finds that the $x=z$ (or $J=\tilde{J}$) plane is closed by RG transformations (i.e. if $x=z$, then $x'=z'$). However, since the fixed point of interest lies outside this region, this fact will not simplify an RG treatment of the general N-state Potts model. Figure 3 shows some typical trajectories in the $x=z$ plane.

On linearising the recursion relations about the fixed point $(0,1,1)$ with $\Delta y = 1-y$ and $\Delta z = 1-z$, one gets

$$x' \sim 2x \qquad \bar{y}' \sim 2\bar{y} \qquad \Delta z' \sim \Delta z \tag{2.6}$$

where $\bar{y} = \Delta y + \Delta z/2$. Thus $\Delta x \sim 3\tilde{K}/2$ is a marginal field. Its corresponding eigenoperator is essentially $\Sigma \cos[2\pi(n_i+n_j)/3]$.

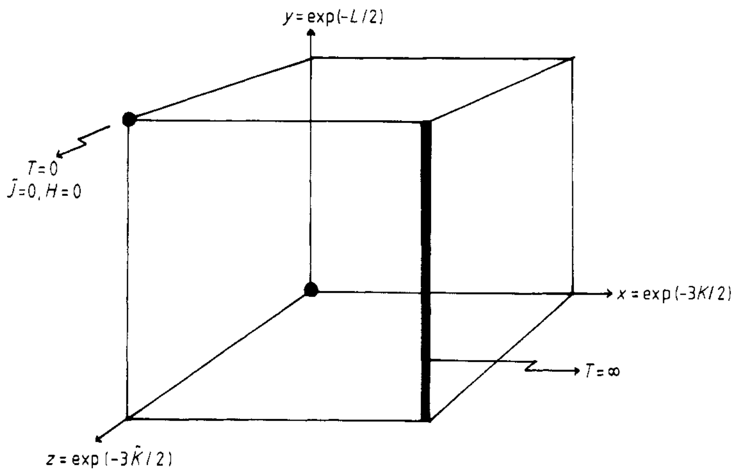

Figure 2. The unit cube $J>0$, $\tilde{J}>0$ in the (x, y, z) space showing the fixed points for the $b=2$ dedecoration group. The line $x^* = z^* = 1$, independent of y, is a continuum of attractive fixed points and corresponds to $T=\infty$ (or $J=\tilde{J}=0$).

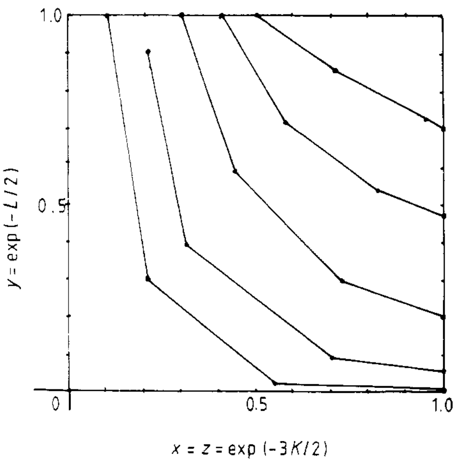

Figure 3. Typical trajectories and fixed points in the $x = z$ plane obtained from the recursion relations (2.2) and (2.3).

Following Nelson and Fisher (1975) one obtains the free energy and correlation function predictions:

$$f(x, \bar{y}, \Delta z) \sim xY(\bar{y}/x, \Delta z) \tag{2.7}$$

$$\sim \exp(-3K/2)\, Y[\exp(3K/2)(L/2+3\tilde{K}/4), 3\tilde{K}/2] \tag{2.8}$$

$$G(R, x, \bar{y}, \Delta z) \sim D(Rx, \bar{y}/x, \Delta z). \tag{2.9}$$

Strictly speaking, to arrive at (2.7) one must first be able to find the dependence on the constant terms appearing in (1.4). But near the critical point (0, 1, 1) we can write

$$(I+J+\tilde{J}+H)' \sim 2(I+J+\tilde{J}+H). \tag{2.10}$$

Now, choosing $I = -J - \tilde{J} - H$ we have $I^{(l)} = -J^{(l)} - \tilde{J}^{(l)} - H^{(l)}$ after l iterations. Therefore the I dependence can be neglected as long as one remains inside the linear region.

By recognising the scaling combination $Rx = R/\xi(T)$ in (2.9) one obtains that the zero-field correlation length diverges as $\exp(3K/2)$ for $\tilde{J} = 0$.

The low temperature properties can be immediately obtained. By differentiating (2.8) twice with respect to L one sees that the zero-field susceptibility diverges exponentially (for $\tilde{J} = 0$) as $\exp(3K/2)$ when $K = J/(K_B T) \to \infty$. Defining the reduced critical exponents in terms of the correlation length $\xi(T)$, i.e.

$$\Delta f = f - f_c \sim \xi^{-(2-\alpha)/\nu} \qquad \chi \sim \xi^{\gamma/\nu} \tag{2.11}$$

one obtains the exponent relations

$$\gamma = \nu \Rightarrow 2 - \alpha. \tag{2.12}$$

The hyperscaling relation $d\nu = 2 - \alpha$, where d stands for the spatial dimensionality, must be compared with the last relation in (2.12). The scaling relation $\gamma = (2-\eta)\nu$ in combination with $\gamma = \nu$ furnishes $\eta = 1$. This result also comes from the renormalisation group identity (see Nelson and Fisher 1975) $d - 2 + \eta = -2 \ln c^*/\ln b$, remembering that $c = 1$, $b = 2$.

All these results for the critical exponents are the same as those already known for the $N = 2$ case.

3. Exact results

One can solve the model described by (1.4) using the transfer matrix formalism. Since this method is well known, we will just show the results without going into the details of the calculations.

The transfer matrix T has the following structure:

$$T=\begin{pmatrix} 1 & B & B \\ B & E & D \\ B & D & E \end{pmatrix} \tag{3.1}$$

where $B=\exp[-\frac{1}{2}(3K+3\tilde{K}+L)]$, $E=\exp(-3\tilde{K}/2-L)$ and $D=\exp(-3K/2-L)$. The eigenvalues are easily found to be

$$\lambda_1=\tfrac{1}{2}\{1+E+D+[(1-E-D)^2+8B^2]^{1/2}\} \tag{3.2}$$

$$\lambda_2=\tfrac{1}{2}\{1+E+D-[(1-E-D)^2+8B^2]^{1/2}\} \tag{3.3}$$

$$\lambda_3=E-D. \tag{3.4}$$

Thus, the free energy per degree of freedom defined by $f\equiv\lim_{N\to\infty}(1/N)\ln Z_N$, where Z_N denotes the partition function, is readily written as

$$F(K,\tilde{K},L)=\ln\lambda_1. \tag{3.5}$$

The matrix U that diagonalises T by a similarity transformation, i.e. $(U^{-1}TU)_{ij}=\lambda_i\delta_{ij}$ is

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix} A_+ & A_- & 0 \\ B_+ & B_- & 1 \\ B_+ & B_- & 1 \end{pmatrix} \tag{3.6}$$

where $A_\pm=-2B[2B^2+(1-\lambda_\pm)^2]^{-1/2}$ and $B_\pm=(1-\lambda_\pm)[2B^2+(1-\lambda_\pm)^2]^{-1/2}$ (here we have changed slightly the notation from λ_1 and λ_2 to λ_+ and λ_-, respectively). Therefore the magnetisation $M\equiv\langle\frac{1}{2}(S_L+S_L^+)\rangle$ can be calculated, resulting in

$$M(K,\tilde{K},L)=\tfrac{1}{2}(A_+^2-B_+^2). \tag{3.7}$$

Figure 4 shows spontaneous magnetisation curves, $M(K,\tilde{K},L=0)$ against $\exp(-3K/2)$ for various values of $\alpha=\tilde{J}/J$. The magnetisation can take on non-zero values at finite temperatures since it is not an order parameter for $\alpha\neq 0$.

The correlation function $G(R,K,\tilde{K},L)\equiv\langle\frac{1}{2}(S_L+S_L^+)\frac{1}{2}(S_{L'}+S_{L'}^+)\rangle$ can also be calculated (see, for example, McCoy and Wu 1973) resulting in

$$G(R,K,\tilde{K},L)=\tfrac{1}{4}[(A_+^2-B_+^2)^2+(\lambda_2/\lambda_1)^R(A_+A_--B_+B_-)^2]. \tag{3.8}$$

Using the fluctuation dissipation theorem (Stanley 1971) one obtains the magnetic susceptibility

$$\chi(T,H)=\frac{1}{4K_BT}\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}(A_+A_--B_+B_-)^2. \tag{3.9}$$

Curves of the zero-field susceptibility in J units against $x=\exp(-3K/2)$ for various values of $\alpha=\tilde{J}/J$ are shown in figure 5. One can see that when $x\to 0$ (for $\alpha>0$), $J\chi_\alpha(x)$ goes to zero as $-x^{2\alpha+2}\ln x$. For $\alpha=0$ (now we must take the lim $\alpha\to 0$ first) the susceptibility diverges as $(-1/x)\ln x$ when $x\to 0$. The divergence in the Ising case $(N=2)$ has the form (Stanley 1971) $(-1/\bar{x})\ln\bar{x}$ with $\bar{x}=\exp(-2K)$. Both $\bar{x}$ and x

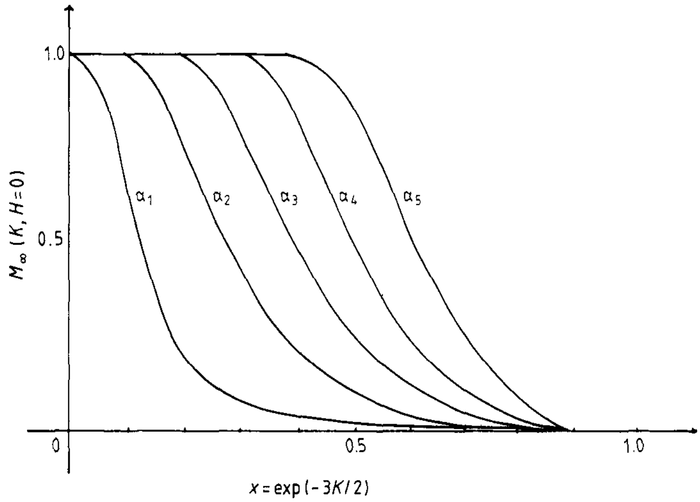

Figure 4. M_α $(K, H=0)$ against $\exp(-3K/2)$ for various values of $\alpha = \tilde{J}/J$: $\alpha_1 = 0{\cdot}1$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = 1$, $\alpha_4 = 2$, $\alpha_5 = 4$. The spontaneous magnetisation is not an order parameter for $\alpha \neq 0$ and so it can take on non-zero values at finite temperatures. As $\alpha \to 0$ (i.e. $\tilde{J} \to 0$) the $Z(3)$ symmetry is recovered and $M \to 0$ for $T \neq 0$, as it should.

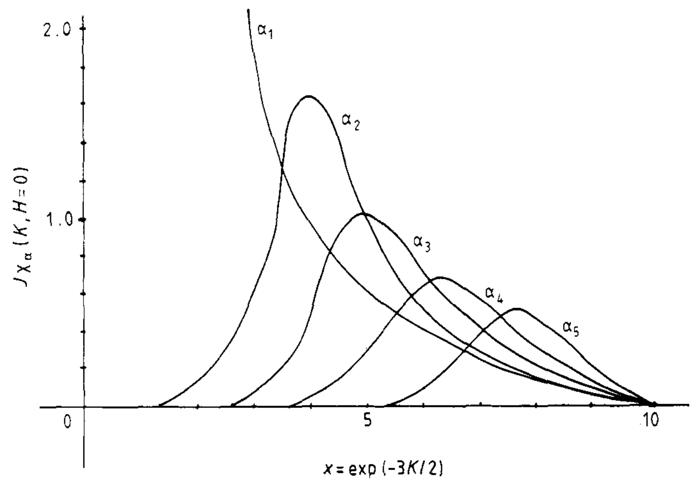

Figure 5. Susceptibility in J units against $x = \exp(-3K/2)$ for various values of $\alpha = \tilde{J}/J$: $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = 1$, $\alpha_4 = 2$, $\alpha_5 = 4$.

are the inverse of the statistical weights of the Ising and 3-state Potts models respectively. One defines the statistical weights of systems displaying the $Z(N)$ symmetry as

$$w_n = \exp \sum_{m=1}^{\tilde{N}} K_m[\cos(2\pi m/N)n - 1](\mathrm{mod}\ N) \qquad n = 1, 2 \ldots N-1.$$

4. Exact results and RG predictions

Near the fixed point (0, 1, 1) the free energy (3.5) can be written as

$$f \sim x\{\tfrac{1}{2} - \bar{y}/x + [(\bar{y}/x)^2 - \bar{y}/x + 9/4]^{1/2}\} \tag{4.1}$$

which confirms the RG prediction (2.7). The prediction for the correlation function also agrees with the exact result. In order to see this we note that, near (0, 1, 1), we get

$$(\lambda_2/\lambda_1)^R \sim (1-2xg)^R \sim 1-2Rxg \tag{4.2}$$

in which $g=[(\bar{y}/x)^2-\bar{y}/x+9/4]^{1/2}$. With respect to the R-independent part of $G(R, K, \tilde{K}, L)$ one has

$$A_\pm = -2[2g^2 \pm g(1-2\bar{y}/x)]^{-1/2} \tag{4.3}$$

and

$$B_\pm = (\bar{y}/x - \tfrac{1}{2} \pm g)[2g^2 \pm g(1-2\bar{y}/x)]^{-1/2}. \tag{4.4}$$

It is easy to see that (4.2), (4.3) and (4.4) confirm the RG prediction (2.9).

5. Conclusions

We focused on the feasibility of the dedecoration group transformation on $Z(N)$ models in a non-zero magnetic field. The zero-field case presents no difficulty. In fact, the N-general model in zero magnetic field is even easier to treat, both exactly and via RG formalism, than the $N=3$ model in a non-zero magnetic field (Cressoni 1981). However, the introduction of a magnetic field is highly desirable in order to obtain, for example, the magnetisation and susceptibility critical behaviour. Therefore, we were led to reinvestigate the problem, taking advantage of the simplicity of the linear chain. In spite of having considered just the $N=3$ case we are confident at the possibility of generalisation to $N>3$ models and more complex lattices presently under investigation.

Acknowledgment

One of us (JCC) would like to thank Professor F C Alcaraz for his helpful conversations.

References

Alcaraz F C and Köberle R 1981 *J. Phys. A: Math. Gen.* **14** 1169-92
Cressoni J C 1981 *Thesis* Instituto de Física e Química de São Carlos
Fisher M E 1959 *Phys. Rev.* **113** 969-81
McCoy B M and Wu T T 1973 *The Two Dimensional Ising Model* (Harvard: Harvard University Press) p 40
Nelson D R and Fisher M E 1975 *Ann. Phys., NY* **91** 226-74
Stanley H E 1971 *Introduction to Phase Transitions and Critical Phenomena* (Oxford: Oxford University Press) p 262